ETMAG

LECTURE 8

Continuous functions:

- Definitions, examples
- Arithmetic properties
- Composition (including proper proof)
- Intermediate value thm.

Definition.

A function $f(x)$ is said to be *continuous at a point* p iff

1. $p \in Dom_f$
2. $\lim_{x \to p} f(x)$ exists
3. $\lim_{x \to p} f(x) = f(p)$

Remark

If $f(x)$ is not continuous at p then p is called *a* (*point of*) *discontinuity* of f.

Definition.

A function $f(x)$ is said to be *continuous on a set* S iff

$f(x)$ is continuous at every point $p \in S$.
If $f(x)$ is continuous on Dom_f then f is just called *continuous*.

Examples.

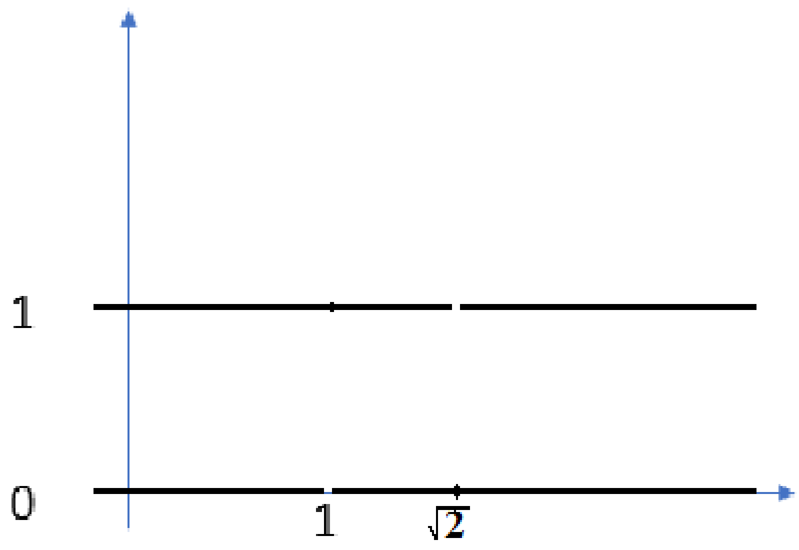
Functions $f(x) = x, \sin x, \tan x$, constant functions, logarithmic functions, exponential functions, polynomial functions are all continuous.

$f(x) = \begin{cases} \frac{|x|}{x} & \text{for } x \neq 0 \\ 0 & \text{for x} = 0 \end{cases}$ is continuous everywhere except at 0.

Examples cont'd.

The *Dirichlet function* $D(x) = \begin{cases} 1 \text{ for } x \in \mathbb{Q} \\ 0 \text{ for x} \notin \mathbb{Q} \end{cases}$ is discontinuous everywhere. In fact $D(x)$ does not have a limit at any point p because for every $\delta > 0$, the interval $(p - \delta, p + \delta)$ contains rational and irrational numbers.

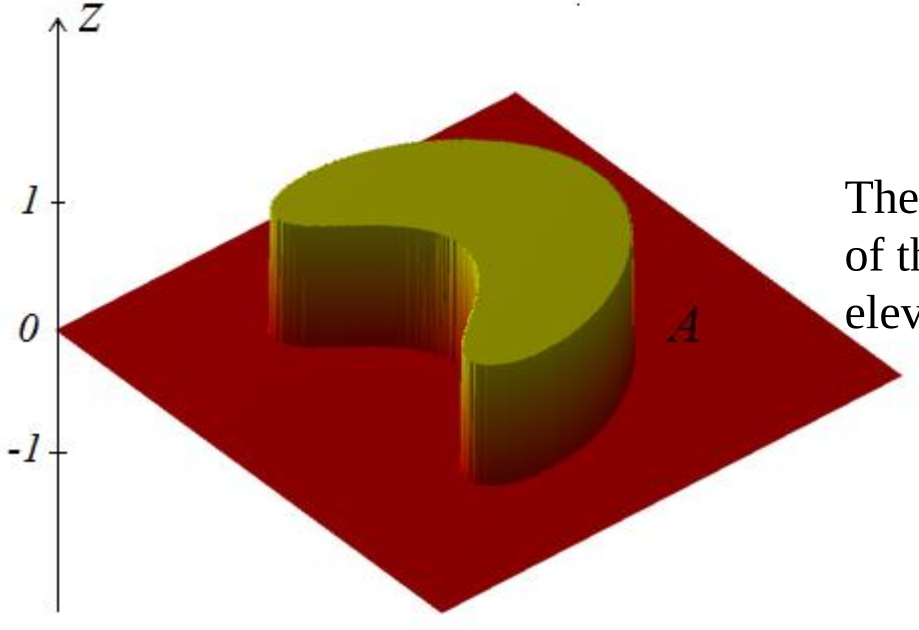
Notice that $D(x)$ is NOT an elementary function.



Plotting the graph of $D(x)$ like this is cheating. In both lines $y = 1$ and $y = 0$ there are infinitely many gaps which can hardly be rendered in a graph.a

Remark.

Given a subset $A \subseteq X$, the function $\mathbb{1}_{\mathrm{A}}(x) = \begin{cases} 1 \text{ for } x \in A \\ 0 \text{ for x} \notin A \end{cases}$ is called *the indicator* (or *characteristic*) *function of* A. In that terminology $D(x)$ is the characteristic function of $\mathbb{Q}$.

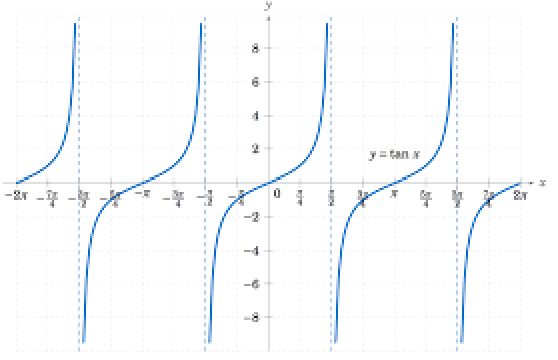


The graph of the indicator function of the plane region A (under the elevated portion of the XY plane).

The *Dirichlet function* is sometimes called *a pathologic function* because of its strange properties. It is the basis for many funny examples. For instance, using $D(x)$ we can construct a function which has the whole set $\mathbb{R}$ as its domain and has exactly one point of continuity. E.g. $f(x) = xD(x)$. Clearly, since $f(0) = 0$ and $\lim_{x \to 0} f(x) = 0$, $x = 0$ is the only point at which $f(x)$ is continuous ($\lim_{x \to 0} f(x) = 0$ by the squeeze theorem).

Remark.

It is wrong to think that continuous functions are those whose graphs look like continuous curves (without gaps). The graph of a continuous function looks like an unbroken curve only if considered **on an interval contained in its domain**. For example, the *signum* function (1 if $x > 0$, -1 if $x < 0$) is continuous in its domain ($\mathbb{R}$ without 0) even though its graph has a discontinuity. For the same reason $tan(x)$ is considered continuous. Even outrageously discontinuous functions, like the Dirichlet function $D(x)$, are continuous if you restrict them to a subset of $\mathbb{R}$**,** for example, it is continuous on $\mathbb{Q}$ (well, it is constant there)**.**



$\tan(x)$ is continuous everywhere in its domain but it is not continuous on $\mathbb{R}$. It is continuous on every open interval which does not contain any point from the set $\{k\pi + \frac{\pi}{2} | k \in \mathbb{Z}\}$ and on each of those the graph of $\tan(x)$ is an unbroken curve.

Theorem. (arithmetic properties of continuous functions)

Let $f(x)$ and $g(x)$ be functions continuous at a point p. Then:

1. $f + g$ is continuous at p,
2. fg is continuous at p,
3. $\frac{f}{g}$ is continuous at p if $g(p) \neq 0$.

Notice that 1. and 2. imply that for every constant c, cf is continuous at p and $f - g$ is continuous at p.

Theorem.
Let $f(x)$ be continuous at a point p and let $g(x)$ be continuous at $q = f(p)$. Then $g \circ f$ is continuous at p.

Finally we have something which is not a straightforward extension of properties of the limit of sequences. Composition of sequences usually makes no sense.

In the Nov. 23 lecture I gave you an outline of the proof only. You will find the full version below.

Proof.
We need to prove that $\lim_{x\to p} g\big(f(x)\big) = g\big(f(p)\big)$ i.e., given $\varepsilon > 0$ we must find $\delta > 0$ such that
$(\forall x)\ 0 < |x - p| < \delta \Rightarrow \left|g\big(f(x)\big) - g\big(f(p)\big)\right| < \varepsilon.$
Since $g(x)$ is continuous at $f(p)$ we have $\lim_{t\to f(p)} g(t) = g\big(f(p)\big)$ which means there exists $\delta_g > 0$, such that
$(*)\ (\forall t)\ 0 < |t - f(p)| < \delta_g \Rightarrow \left|g(t) - g\big(f(p)\big)\right| < \varepsilon.$
Since $\lim_{x\to p} f(x) = f(p)$ there exists $\delta_f > 0$ such that
$(**)(\forall x)\ 0 < |x - p| < \delta_f \Rightarrow |f(x) - f(p)| < \delta_g$ (yes, δ_g from $(*)$).
This means if $0 < |x - p| < \delta_f$, $f(x)$ can be used as t in $(*)$.
Concluding, δ_f can serve as the δ we are looking for. QED

Remark.
The last theorem means that the composition of two continuous functions is itself continuous – if we choose carefully the points of continuity.

That is something we didn't and couldn't say about sequences.

Corollary.
All elementary functions are continuous.

Theorem. (Intermediate Value Theorem, Darboux Theorem)
If a function f is continuous on a closed interval $[a, b]$ then f takes on every value between $f(a)$ and $f(b)$.

To be more precise: if f is continuous on $[a;\ b]$ then for every $y_0 \in [f(a), f(b)]$ (or $y_0 \in [f(b), f(a)]$, if $f(b) \leq f(a)$) there exists $x_0 \in [a, b]$ such that $f(x_0) = y_0$.

The proof is beyond the scope of this course.

Corollary.
If f is continuous on $[a, b]$ and $f(a)$ and $f(b)$ differ in sign then there exists at least one $x \in [a, b]$ such that $f(x) = 0$.

Corollary. (of corollary)
Every polynomial of an odd degree has at least one (real) root.